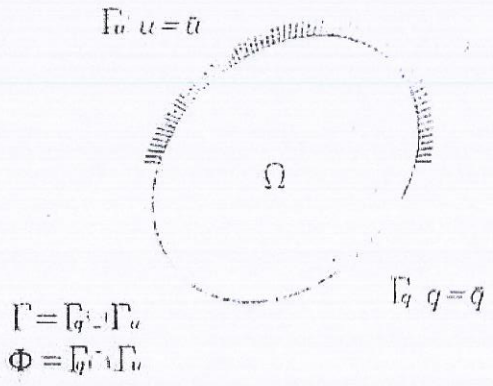
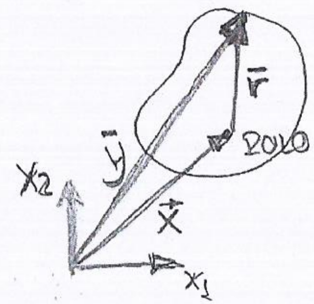
Problemas de potencial: Determinación de una función armónica u, en el dominio Ω, cuya frontera es Γ; estando sometido el dominio Ω a condiciones de contorno que pueden ser esenciales o naturales; utilizando para el cálculo de u, la deducida fórmula de Green:

$$\nabla^2 u = 0 \quad \forall \mathbf{x} \in \Omega$$



$$\begin{cases} u = \bar{u} \quad \forall \mathbf{x} \in \Gamma_u \\ q = \bar{q} \quad \forall \mathbf{x} \in \Gamma_q \\ \Gamma = \Gamma_u \cup \Gamma_q \\ \Phi = \Gamma_u \cap \Gamma_q \end{cases}$$

$$u(\mathbf{x}) = \int_\Gamma (u^*(\mathbf{y};\mathbf{x})q(\mathbf{y}) - u(\mathbf{y})q^*(\mathbf{y};\mathbf{x}))d\Gamma(\mathbf{y})$$



siendo,

$u(\mathbf{y})$: función incógnita; $u(\mathbf{x})$: valor de la función u en el punto $\mathbf{x}$ $\left\{ \begin{array}{l} \mathbf{x} \in \Omega \\ \mathbf{x} \notin \Gamma \end{array} \right\}$ que llamamos polo.

$q(\mathbf{y}) = \dfrac{\partial u}{\partial \hat{\mathbf{n}}}$; $\quad \hat{\mathbf{n}}$:normal exterior a Γ

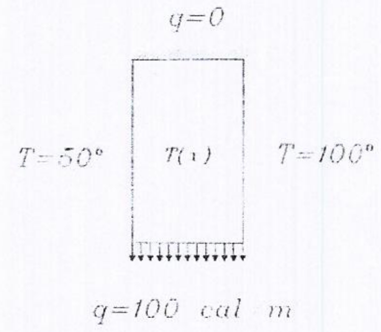
$u^*(\mathbf{y};\mathbf{x}) = \dfrac{1}{2\pi} ln\left\{\dfrac{1}{r}\right\}$; $\quad r = |\mathbf{y} - \mathbf{x}|$

$$q^*(\mathbf{y};\mathbf{x}) = \frac{\partial u^*}{\partial \hat{\mathbf{n}}} = -\frac{1}{2\pi r}\frac{\partial r}{\partial \hat{\mathbf{n}}}$$

$$u^* = \frac{1}{2\pi} Ln\left(\frac{1}{r}\right) \qquad \frac{\partial}{\partial x_i}\left\{Ln\left(\frac{1}{r}\right)\right\} = \frac{-x_i}{r^2} \qquad \vec{\nabla} u^* = \frac{-1}{2\pi r^2}\vec{r} \quad \vec{\nabla} r = \frac{1}{r}\vec{r}$$

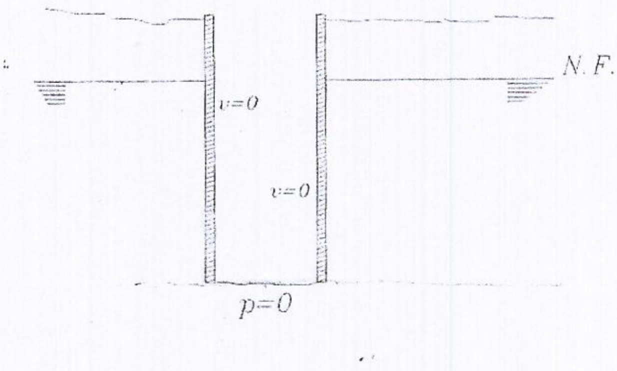
$$q^* = \frac{\partial u^*}{\partial \hat{n}} = \vec{\nabla} u^* \cdot \hat{n} = \frac{-1}{2\pi r^2}\vec{r}\cdot\hat{n} = \frac{-1}{2\pi r^2} r\vec{\nabla} r \cdot \hat{n} = \frac{-1}{2\pi r}\frac{\partial r}{\partial \hat{n}}$$

Problemas de distribución de temperatura:



$$\left\{ \begin{array}{l} T : \textit{temperatura} \\ \\ q : \textit{flujo de calor} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \nabla^2 T = 0 \\ q = -\lambda \frac{\partial T}{\partial \hat{\mathbf{n}}} \\ \lambda : \textit{conductividad térmica} \end{array} \right.$$

Presión hidrostática en el terreno:



$$\textit{Potencial} : u = p + z : \left\{ \begin{array}{l} p : \textit{presión de agua en el terreno} \\ z : \textit{altura} \end{array} \right.$$

$\textit{Ley de Darcy} : \mathbf{v} = -k\nabla u$

$\textit{Movimiento isovolumen} : \nabla \bullet \mathbf{v} = 0 \Rightarrow -k\nabla^2 u = 0 \Rightarrow \nabla^2 u = 0$

$\textit{Caudal de fluido} : \ v = \mathbf{v} \bullet \hat{\mathbf{n}} = -k\nabla u \bullet \hat{\mathbf{n}} = -k\frac{\partial u}{\partial \hat{\mathbf{n}}} \quad (m^3/m^2)$

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1ª Identidad de Green: aplicando el Teorema de la Divergencia a $u\boldsymbol{\nabla}w$:

$$\int_\Omega (\boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}w + u\nabla^2 w)\, d\Omega = \int_\Gamma u\frac{\partial w}{\partial \hat{n}}\, d\Gamma$$

2ª Identidad de Green: aplicando 1ª Identidad a w,u:

$$\int_\Omega (u\nabla^2 w - w\nabla^2 u)\, d\Omega = \int_\Gamma \left(u\frac{\partial w}{\partial \hat{n}} - w\frac{\partial u}{\partial \hat{n}}\right) d\Gamma$$

3ª Identidad de Green: aplicando 2ª Identidad a $u, u^* = \frac{1}{2\pi}Ln\left(\frac{1}{r}\right)$:

$$-\int_\Omega \frac{1}{2\pi} ln\left\{\frac{1}{r}\right\}\nabla^2 u\, d\Omega = \int_\Gamma \left(u\frac{\partial}{\partial \hat{n}}\left(\frac{1}{2\pi}ln\left(\frac{1}{r}\right)\right) - \frac{1}{2\pi}ln\left(\frac{1}{r}\right)\frac{\partial u}{\partial \hat{n}}\right) d\Gamma$$

Si el polo es interior al dominio:

$$-\int_\Omega \frac{1}{2\pi} ln\left\{\frac{1}{r}\right\}\nabla^2 u\, d\Omega = \int_\Gamma \left(u\frac{\partial}{\partial \hat{n}}\left(\frac{1}{2\pi}ln\left(\frac{1}{r}\right)\right) - \frac{1}{2\pi}ln\left(\frac{1}{r}\right)\frac{\partial u}{\partial \hat{n}}\right) d\Gamma + u(\mathbf{x})$$

Fórmula de Green: aplicando 3ª Identidad a una función u armónica:

$$u(\boldsymbol{x}) = \int_\Gamma \left(u^*\frac{\partial u}{\partial \hat{\boldsymbol{n}}} - u\frac{\partial u^*}{\partial \hat{\boldsymbol{n}}}\right) d\Gamma \qquad u(\boldsymbol{x}) = \int_\Gamma \left(\frac{1}{2\pi}ln\left(\frac{1}{r}\right)\frac{\partial u}{\partial \hat{\boldsymbol{n}}} - u\frac{\partial}{\partial \hat{\boldsymbol{n}}}\left(\frac{1}{2\pi}ln\left(\frac{1}{r}\right)\right)\right) d\Gamma$$

Estudio de la función ln(1/r) en dominios bidimensionales.

Si establecemos el origen del sistema de referencia OX_1X_2 en O podemos expresar,

$$r = \sqrt{x_1^2 + x_2^2} \Rightarrow \begin{cases} \frac{\partial r}{\partial x_i} = \frac{x_i}{r} \\ \frac{\partial}{\partial x_i}\left(\ln\left\{\frac{1}{r}\right\}\right) = \frac{-x_i}{r^2} \\ \frac{\partial^2}{\partial x_i^2}\left(\ln\left\{\frac{1}{r}\right\}\right) = \frac{2x_i^2 - r^2}{r^4} \end{cases}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

teniendo en cuenta las expresiones anteriores podemos expresar,

$$\nabla^2\left(\ln\left\{\frac{1}{r}\right\}\right) = \frac{2r^2 - 2r^2}{r^4} = 0$$

salvo cuando $r = 0$, que su valor es indefinido.

CONCLUSION

Las expresiones obtenidas indican que la solución fundamental verifica la ecuación de Laplace, salvo en el *polo*, $\mathbf{x}$ $(r = 0)$, que es un punto singular,

$$\nabla^2 u^* = 0; \quad \text{salvo en el punto} \quad r = 0$$

Tercera identidad de Green. Dominios bidimensionales.

Aplicando la segunda identidad de Green a las funciones u y u^* obtenemos la tercera identidad de Green.

$$-\int_\Omega \frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\nabla^2 u d\Omega = \int_\Gamma \left(u\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\right) - \frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial \mathbf{n}}\right) d\Gamma$$

Distinguimos dos casos:

- El polo es exterior a Ω : la función $\ln\left\{\frac{1}{r}\right\}$ no presenta singularidades y por lo tanto la expresion obtenida es válida.
- El polo es interior a Ω : la función $\ln\left\{\frac{1}{r}\right\}$ presenta una singularidad en el polo, y la expresión no está definida. Para hallar su valor efectuamos una operación de paso al límite.

Consideramos un subdominio circular infinitesimal Ω_ε en el entorno del polo. La función $\ln\left\{\frac{1}{r}\right\}$ no presenta puntos de singularidad en el dominio reducido $\Omega - \Omega_\varepsilon$, por lo que podemos plantear la segunda identidad de Green. El contorno del dominio reducido será el contorno ampliado $\Gamma + \Gamma_\varepsilon$. Mediante una operación de paso al límite obtendremos la solución de nuestro problema,

$$\lim_{\varepsilon\to 0}\left\{\int_{\Omega-\Omega_\varepsilon}\left(-\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\nabla^2 u\right) d\Omega\right\} = \lim_{\varepsilon\to 0}\left\{\int_{\Gamma+\Gamma_\varepsilon}\left(u\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\right) - \frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial \mathbf{n}}\right) d\Gamma\right\}$$

Calculamos los límites aparecidos en la expresión anterior:

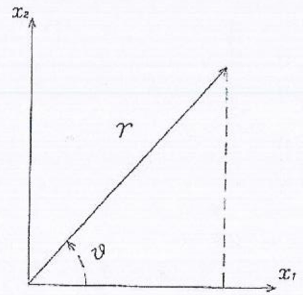
- $\lim_{\varepsilon\to 0}\left\{\int_{\Omega-\Omega_\varepsilon}\left(-\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\nabla^2 u\right) d\Omega\right\} = \int_\Omega\left(-\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\nabla^2 u\right) d\Omega$
- $\lim_{\varepsilon\to 0}\left\{\int_\Gamma\left(u\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\right) - \frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial \mathbf{n}}\right) d\Gamma\right\} = \int_\Gamma\left(u\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\right) - \frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial \mathbf{n}}\right) d\Gamma$
- $\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon}\left(u\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\right) - \frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial \mathbf{n}}\right) d\Gamma\right\} = u(\mathbf{x})$: vemos con detalle.

En los puntos de Γ_ε podemos expresar

$$\frac{\partial r}{\partial \mathbf{n}} = \boldsymbol{\nabla} r \cdot \mathbf{n} = \frac{1}{r}\mathbf{r}\cdot\mathbf{n} = -1 \Rightarrow \frac{\partial}{\partial \mathbf{n}}\left(\ln\left\{\frac{1}{r}\right\}\right) = \frac{\partial}{\partial r}\left(\ln\left\{\frac{1}{r}\right\}\right)\frac{\partial r}{\partial \mathbf{n}} = -\frac{\partial}{\partial r}\left(\ln\left\{\frac{1}{r}\right\}\right) = \frac{1}{r}$$

$$\int_{\Gamma_\varepsilon}\left(u\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\right) - \frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial \mathbf{n}}\right)d\Gamma = \frac{1}{2\pi}\int_{\Gamma_\varepsilon}\left(u\left(\frac{1}{r}\right) - \ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial \mathbf{n}}\right)d\Gamma =$$

Expresando la integral en coordenadas polares, $d\Gamma = r d\theta$



$$= \frac{1}{2\pi}\int_{\Gamma_\varepsilon}\left(u\left(\frac{1}{\varepsilon}\right) - \ln\left\{\frac{1}{\varepsilon}\right\}\frac{\partial u}{\partial \mathbf{n}}\right)\varepsilon d\theta = \frac{1}{2\pi}\int_{\Gamma_\varepsilon} u\,d\theta - \frac{1}{2\pi}\int_{\Gamma_\varepsilon}\varepsilon \ln\left\{\frac{1}{\varepsilon}\right\}\frac{\partial u}{\partial \hat{n}}\,d\theta$$

$$\frac{1}{2\pi}\underbrace{\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} u\,d\theta}_{2\pi u(\bar{x})} - \frac{1}{2\pi}\underbrace{\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\varepsilon \ln\left\{\frac{1}{\varepsilon}\right\}\frac{\partial u}{\partial \hat{n}}\,d\theta}_{=\,0}$$

- Dado el carácter infinitésimal de Γ_ε, podemos considerar $u \simeq u(\mathbf{x})$, por lo tanto,

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} u d\theta\right\} = 2\pi u(\mathbf{x})$$

- Supuesta la función $\frac{\partial u}{\partial \mathbf{n}}$ regular en el dominio Γ_ε, la expresión $\int\int_{\Gamma_\varepsilon}\frac{\partial u}{\partial \mathbf{n}}d\theta$ será igual a un valor finito, por lo que

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon}\ln\left\{\frac{1}{\varepsilon}\right\}\frac{\partial u}{\partial \mathbf{n}}\varepsilon d\theta\right\} = \lim_{\varepsilon\to 0}\left\{\varepsilon\ln\left\{\frac{1}{\varepsilon}\right\}\int_{\Gamma_\varepsilon}\frac{\partial u}{\partial \mathbf{n}}d\theta\right\} = 0$$

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Tercera identidad de Green. Dominios bidimensionales.

Distinguimos dos casos:

- El polo es exterior a Ω :

$$-\int_{\Omega}\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\nabla^2 u d\Omega=\int_{\Gamma}\left(u\frac{\partial}{\partial\mathbf{n}}\left(\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\right)-\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial\mathbf{n}}\right)d\Gamma$$

- El polo es interior a Ω :

$$-\int_{\Omega}\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\nabla^2 u d\Omega=\int_{\Gamma}\left(u\frac{\partial}{\partial\mathbf{n}}\left(\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\right)-\frac{1}{2\pi}\ln\left\{\frac{1}{r}\right\}\frac{\partial u}{\partial\mathbf{n}}\right)d\Gamma+u\left(\mathbf{x}\right)$$

Tercera identidad de Green. Expresión genérica.

- El polo es exterior a Ω :

$$-\int_{\Omega}u^*\nabla^2 u d\Omega=\int_{\Gamma}\left(u\frac{\partial u^*}{\partial\mathbf{n}}-u^*\frac{\partial u}{\partial\mathbf{n}}\right)d\Gamma$$

- El polo es interior a Ω :

$$-\int_{\Omega}u^*\nabla^2 u d\Omega=\int_{\Gamma}\left(u\frac{\partial u^*}{\partial\mathbf{n}}-u^*\frac{\partial u}{\partial\mathbf{n}}\right)d\Gamma+u\left(\mathbf{x}\right)$$

Fórmula de Green.

Se denomina *fórmula de Green* a la expresión resultante de aplicar la tercera identidad de Green a una función armónica ($\nabla^2 u = 0$) :

- El polo es exterior a Ω :

$$\int_{\Gamma}\left(u\frac{\partial u^*}{\partial\mathbf{n}}-u^*\frac{\partial u}{\partial\mathbf{n}}\right)d\Gamma=0$$

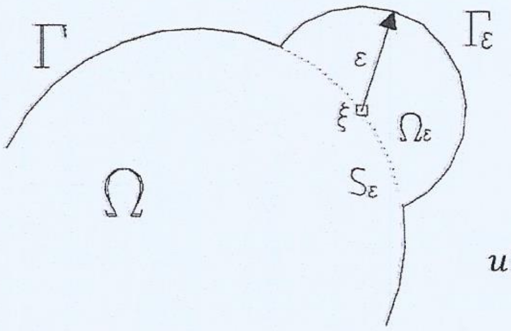
- El polo es interior a Ω :

$$u(\mathbf{x})=\int_{\Gamma}\left(u^*\frac{\partial u}{\partial\mathbf{n}}-u\frac{\partial u^*}{\partial\mathbf{n}}\right)d\Gamma$$

Ecuación integral en el contorno:

$$u(\boldsymbol{x}) = \int_{\Gamma} \left(u^*(\mathbf{y};\mathbf{x})q(\mathbf{y}) - u(\mathbf{y})q^*(\mathbf{y};\mathbf{x})\right) d\Gamma\,(\mathbf{y})$$

$\mathbf{x} \to \boldsymbol{\xi} \in \Gamma$: **Se considera operación de paso al límite en el dominio ampliado**



$$\tilde{\Omega} = \Omega + \Omega_\varepsilon$$
$$\tilde{\Gamma} = \Gamma - S_\varepsilon + \Gamma_\varepsilon$$

$$u(\boldsymbol{\xi}) = \lim_{\varepsilon \to 0}\left\{\int_{\Gamma - S_\varepsilon + \Gamma_\varepsilon} \left(u^*(\mathbf{y};\mathbf{x})q(\mathbf{y}) - u(\mathbf{y})q^*(\mathbf{y};\mathbf{x})\right)d\Gamma(\mathbf{y})\right\}$$

$$u(\xi) = lim_{\varepsilon\to 0}\left\{\int_{\Gamma - S_\varepsilon} u^*(y;x)q(y)d\Gamma(y)\right\} - lim_{\varepsilon\to 0}\left\{\int_{\Gamma - S_\varepsilon} u(y)q^*(y;x)d\Gamma(y)\right\}$$
$$+\ lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} u^*(y;x)q(y)d\Gamma(y)\right\} - lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} u(y)q^*(y;x)d\Gamma(y)\right\}$$

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma - S_\varepsilon} u^* q d\Gamma\right\} = \lim_{\varepsilon\to 0}\left\{\int_{\Gamma - S_\varepsilon} \frac{1}{2\pi} ln\left(\frac{1}{r}\right) q d\Gamma\right\} = \int_{\Gamma} \frac{1}{2\pi} ln\left(\frac{1}{r}\right) q d\Gamma : impropia\ convergente$$

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma - S_\varepsilon} u q^* d\Gamma\right\} = \lim_{\varepsilon\to 0}\left\{\int_{\Gamma - S_\varepsilon} u \frac{-1}{2\pi r}\frac{\partial r}{\partial \hat{n}} d\Gamma\right\} = VPC\int_{\Gamma} u \frac{-1}{2\pi r}\frac{\partial r}{\partial \hat{n}} d\Gamma$$

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} u^* q d\Gamma\right\} = \lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} \frac{1}{2\pi} ln\left(\frac{1}{r}\right) q d\Gamma\right\} = 0$$

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} u q^* d\Gamma\right\} = \lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} u \frac{-1}{2\pi r}\frac{\partial r}{\partial \hat{n}} d\Gamma\right\} = -\frac{1}{2}u(\boldsymbol{\xi})$$

$$\frac{1}{2}u(\xi) = \int_{\Gamma} \frac{1}{2\pi} ln\left(\frac{1}{r}\right) q d\Gamma - VPC\int_{\Gamma} \frac{-1}{2\pi r}\frac{\partial r}{\partial \hat{n}} u d\Gamma$$

$$\boxed{\frac{1}{2}u(\xi) = \int_{\Gamma} u^*(y;\xi)q(y)d\Gamma(y) - VPC\int_{\Gamma} q^*(y;\xi)u(\mathbf{y})d\Gamma(y)}$$

$$\frac{\theta(\xi)}{2\pi}u(\xi) = \int_{\Gamma} u^*(y;\xi)q(y)d\Gamma(y) - VPC\int_{\Gamma} u(y)q^*(y;\xi)d\Gamma(y)$$

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